Bicrossproduct structure of the null-plane quantum Poincaré algebra

Oscar Arratia [1], Francisco J. Herranz [2] and Mariano A. del Olmo [3]

[1] *Departamento de Matemática Aplicada a la Ingeniería,*
Universidad de Valladolid. E-47011, Valladolid, Spain
E. mail: oscarr@wmatem.eis.uva.es

[2] *Departamento de Física, E.U. Politécnica,*
Universidad de Burgos. E-09006, Burgos, Spain
E. mail: fteorica@cpd.uva.es

[3] *Departamento de Física Teórica,*
Universidad de Valladolid. E-47011, Valladolid, Spain
E. mail: olmo@cpd.uva.es

Abstract

A nonlinear change of basis allows to show that the non-standard quantum deformation of the (3+1) Poincaré algebra has a bicrossproduct structure. Quantum universal R-matrix, Pauli–Lubanski and mass operators are presented in the new basis.

1.- The aim of this letter is to prove that the non-standard quantum deformation of the (3+1) Poincaré algebra [1], the so-called null-plane quantum Poincaré algebra, can be endowed with a structure of bicrossproduct [2]. After the proofs by Majid and Ruegg [3] that the (3+1) κ-Poincaré algebra [4, 5, 6] has a bicrossproduct structure, and more recently by Azcárraga *et al* [7] that the q-Poincaré in any dimension [8] has also this kind of structure, it only remains to study if the same bicrossproduct structure is exhibited by the (3+1) null-plane quantum Poincaré. In [9] it has been showed that the (1+1) null-plane quantum Poincaré [10] also shares this structure, however, this lower dimensional case does not indicate the procedure for the (3+1) case, i.e., the nonlinear change of basis that allows to display the bicrossproduct structure. It is worthy to note that in all the three mentioned deformations the formal decomposition is the same, i.e.,

$$U_q(\mathcal{P}(3+1)) = U(so(3,1))^{\beta}\triangleright\!\!\blacktriangleleft_{\alpha} U_q(\mathcal{T}_4),$$

following the same pattern of the classical algebra or group counterpart

$$P(3+1) = SO(3,1) \odot T_4,$$

and with the sector of the translations deformed (differently in each case, of course) and the Lorentz transformation sector non-deformed.

2.- The generators of the (3+1) Poincaré algebra $\mathcal{P}(3+1)$ in the so-called null-plane basis [11] are

$$\{P_+, P_-, P_i, E_i, F_i, K_3, J_3;\ i = 1, 2\}, \tag{1}$$

where P_+, P_-, E_i and F_i are expressed in terms of the usual kinematical ones $\{H, P_l, K_l, J_l;\ l = 1, 2, 3\}$ by

$$\begin{array}{lll} P_+ = (H + P_3)/2, & P_- = H - P_3, & E_1 = (K_1 + J_2)/2, \\ F_1 = K_1 - J_2, & F_2 = K_2 + J_1, & E_2 = (K_2 - J_1)/2. \end{array} \tag{2}$$

Hence, the Lie brackets of $\mathcal{P}(3+1)$ are (hereafter $i, j = 1, 2$):

$$\begin{array}{lll} [K_3, E_i] = E_i, & [K_3, F_i] = -F_i, & [K_3, J_3] = 0, \\ [J_3, E_i] = -\varepsilon_{ij3}E_j, & [J_3, F_i] = -\varepsilon_{ij3}F_j, & [E_1, E_2] = 0, \\ [E_i, F_j] = \delta_{ij}K_3 + \varepsilon_{ij3}J_3, & [F_1, F_2] = 0, & \end{array} \tag{3}$$

$$[P_\mu, P_\nu] = 0, \qquad \mu, \nu = +, -, 1, 2, \tag{4}$$

$$\begin{array}{lll} [K_3, P_+] = P_+, & [K_3, P_-] = -P_-, & [K_3, P_i] = 0, \\ [J_3, P_i] = -\varepsilon_{ij3}P_j, & [J_3, P_+] = 0, & [J_3, P_-] = 0, \\ [E_i, P_j] = \delta_{ij}P_+, & [E_i, P_-] = P_i, & [E_i, P_+] = 0, \\ [F_i, P_j] = \delta_{ij}P_-, & [F_i, P_+] = P_i, & [F_i, P_-] = 0, \end{array} \tag{5}$$

where ε_{ijk} is the completely skewsymmetric tensor.

The semidirect product structure of the (3+1) Poincaré group, isomorphic to $ISO(3,1)$, can be clearly pointed out. The six generators $\{E_i, F_i, K_3, J_3\}$ close the

Lorentz subgroup $SO(3,1)$ (3), while the four remaining $\{P_+, P_-, P_i\}$ generate the abelian subgroup T_4 (4). Therefore, as it is well known, $ISO(3,1) = SO(3,1) \odot T_4$.

3.- A triangular or non-standard quantum deformation of $\mathcal{P}(3+1)$ was introduced in [1] in the null-plane framework above mentioned, whose Hopf structure we rewrite here for sake of completeness and to clarify our main result. Let us denote the null-plane generators X displayed in (1), by $\widetilde{X}$, and by $\tilde{z}$ the deformation parameter.

Coproduct:

$$
\begin{aligned}
&\Delta(\widetilde{X}) = 1 \otimes \widetilde{X} + \widetilde{X} \otimes 1, \qquad \text{for} \quad \widetilde{X} \in \{\widetilde{P}_+, \widetilde{E}_i, \widetilde{J}_3\},\\
&\Delta(\widetilde{Y}) = e^{-\tilde{z}\widetilde{P}_+} \otimes \widetilde{Y} + \widetilde{Y} \otimes e^{\tilde{z}\widetilde{P}_+}, \quad \text{for} \quad \widetilde{Y} \in \{\widetilde{P}_-, \widetilde{P}_i\},\\
&\Delta(\widetilde{F}_1) = e^{-\tilde{z}\widetilde{P}_+} \otimes \widetilde{F}_1 + \widetilde{F}_1 \otimes e^{\tilde{z}\widetilde{P}_+} + \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{E}_1 \otimes \widetilde{P}_- - \tilde{z}\widetilde{P}_- \otimes \widetilde{E}_1 e^{\tilde{z}\widetilde{P}_+}\\
&\qquad\quad + \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{J}_3 \otimes \widetilde{P}_2 - \tilde{z}\widetilde{P}_2 \otimes \widetilde{J}_3 e^{\tilde{z}\widetilde{P}_+},\\
&\Delta(\widetilde{F}_2) = e^{-\tilde{z}\widetilde{P}_+} \otimes \widetilde{F}_2 + \widetilde{F}_2 \otimes e^{\tilde{z}\widetilde{P}_+} + \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{E}_2 \otimes \widetilde{P}_- - \tilde{z}\widetilde{P}_- \otimes \widetilde{E}_2 e^{\tilde{z}\widetilde{P}_+}\\
&\qquad\quad - \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{J}_3 \otimes \widetilde{P}_1 + \tilde{z}\widetilde{P}_1 \otimes \widetilde{J}_3 e^{\tilde{z}\widetilde{P}_+},\\
&\Delta(\widetilde{K}_3) = e^{-\tilde{z}\widetilde{P}_+} \otimes \widetilde{K}_3 + \widetilde{K}_3 \otimes e^{\tilde{z}\widetilde{P}_+} + \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{E}_1 \otimes \widetilde{P}_1 - \tilde{z}\widetilde{P}_1 \otimes \widetilde{E}_1 e^{\tilde{z}\widetilde{P}_+}\\
&\qquad\quad + \tilde{z}e^{-\tilde{z}\widetilde{P}_+}\widetilde{E}_2 \otimes \widetilde{P}_2 - \tilde{z}\widetilde{P}_2 \otimes \widetilde{E}_2 e^{\tilde{z}\widetilde{P}_+};
\end{aligned} \tag{6}
$$

Counit and Antipode:

$$
\epsilon(\widetilde{X}) = 0; \quad \gamma(\widetilde{X}) = -e^{3\tilde{z}\widetilde{P}_+}\ \widetilde{X}\ e^{-3\tilde{z}\widetilde{P}_+}, \quad \text{for } \widetilde{X} \in \{\widetilde{P}_\pm, \widetilde{P}_i, \widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_3, \widetilde{J}_3\}; \tag{7}
$$

Non-vanishing Lie brackets:

$$
\begin{aligned}
&[\widetilde{K}_3, \widetilde{P}_+] = \frac{\sinh \tilde{z}\widetilde{P}_+}{\tilde{z}}, \quad [\widetilde{K}_3, \widetilde{P}_-] = -\widetilde{P}_- \cosh \tilde{z}\widetilde{P}_+, \quad [\widetilde{K}_3, \widetilde{E}_i] = \widetilde{E}_i \cosh \tilde{z}\widetilde{P}_+,\\
&[\widetilde{K}_3, \widetilde{F}_1] = -\widetilde{F}_1 \cosh \tilde{z}\widetilde{P}_+ + \tilde{z}\widetilde{E}_1\widetilde{P}_- \sinh \tilde{z}\widetilde{P}_+ - \tilde{z}^2\widetilde{P}_2\,\widetilde{W}^z_+,\\
&[\widetilde{K}_3, \widetilde{F}_2] = -\widetilde{F}_2 \cosh \tilde{z}\widetilde{P}_+ + \tilde{z}\widetilde{E}_2\widetilde{P}_- \sinh \tilde{z}\widetilde{P}_+ + \tilde{z}^2\widetilde{P}_1\,\widetilde{W}^z_+,\\
&[\widetilde{J}_3, \widetilde{P}_i] = -\varepsilon_{ij3}\widetilde{P}_j, \qquad [\widetilde{J}_3, \widetilde{E}_i] = -\varepsilon_{ij3}\widetilde{E}_j, \qquad [\widetilde{J}_3, \widetilde{F}_i] = -\varepsilon_{ij3}\widetilde{F}_j,\\
&[\widetilde{E}_i, \widetilde{P}_j] = \delta_{ij}\frac{\sinh \tilde{z}\widetilde{P}_+}{\tilde{z}}, \qquad [\widetilde{F}_i, \widetilde{P}_j] = \delta_{ij}\widetilde{P}_- \cosh \tilde{z}\widetilde{P}_+,\\
&[\widetilde{E}_i, \widetilde{F}_j] = \delta_{ij}\widetilde{K}_3 + \varepsilon_{ij3}\widetilde{J}_3 \cosh \tilde{z}\widetilde{P}_+, \qquad [\widetilde{P}_+, \widetilde{F}_i] = -\widetilde{P}_i,\\
&[\widetilde{F}_1, \widetilde{F}_2] = \tilde{z}^2\widetilde{P}_-\widetilde{W}^z_+ + \tilde{z}\widetilde{P}_-\widetilde{J}_3 \sinh \tilde{z}\widetilde{P}_+, \qquad [\widetilde{P}_-, \widetilde{E}_i] = -\widetilde{P}_i,
\end{aligned} \tag{8}
$$

where $\widetilde{W}^z_+$ is a component of the deformed Pauli–Lubanski vector defined as

$$
\widetilde{W}^z_+ = \widetilde{E}_1\widetilde{P}_2 - \widetilde{E}_2\widetilde{P}_1 + \widetilde{J}_3\frac{\sinh \tilde{z}\widetilde{P}_+}{\tilde{z}}. \tag{9}
$$

4.- In the sequel we show that this quantum algebra has a bicrossproduct structure [2]. Let us consider the map defined by:

$$
\begin{aligned}
&P_+ = \widetilde{P}_+, \qquad E_i = \widetilde{E}_i, \qquad J_3 = \widetilde{J}_3, \qquad z = 2\tilde{z},\\
&P_- = e^{-\tilde{z}\widetilde{P}_+}\widetilde{P}_-, \qquad P_i = e^{-\tilde{z}\widetilde{P}_+}\widetilde{P}_i,
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= e^{-\tilde{z}\widetilde{P}_+}(\widetilde{F}_1 - \tilde{z}\widetilde{E}_1\widetilde{P}_- - \tilde{z}\widetilde{J}_3\widetilde{P}_2),\\
F_2 &= e^{-\tilde{z}\widetilde{P}_+}(\widetilde{F}_2 - \tilde{z}\widetilde{E}_2\widetilde{P}_- + \tilde{z}\widetilde{J}_3\widetilde{P}_1),\\
K_3 &= e^{-\tilde{z}\widetilde{P}_+}(\widetilde{K}_3 - \tilde{z}\widetilde{E}_1\widetilde{P}_1 - \tilde{z}\widetilde{E}_2\widetilde{P}_2).
\end{aligned} \tag{10}
$$

By applying (10) to the Hopf algebra $U_{\tilde{z}}(\mathcal{P}(3+1))$, whose relations appear displayed in expressions (6)–(8), we get the Hopf algebra $U_z(\mathcal{P}(3+1))$, characterized by the following coproduct, counit, antipode and commutation relations:

$$
\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1, \quad X \in \{P_+, E_i, J_3\},\\
&\Delta(Y) = e^{-zP_+} \otimes Y + Y \otimes 1, \qquad Y \in \{P_-, P_i\},\\
&\Delta(F_1) = e^{-zP_+} \otimes F_1 + F_1 \otimes 1 - zP_- \otimes E_1 - zP_2 \otimes J_3,\\
&\Delta(F_2) = e^{-zP_+} \otimes F_2 + F_2 \otimes 1 - zP_- \otimes E_2 + zP_1 \otimes J_3,\\
&\Delta(K_3) = e^{-zP_+} \otimes K_3 + K_3 \otimes 1 - zP_1 \otimes E_1 - zP_2 \otimes E_2;
\end{aligned} \tag{11}
$$

$$
\epsilon(X) = 0, \qquad X \in \{P_\pm, P_i, E_i, F_i, K_3, J_3\}; \tag{12}
$$

$$
\begin{aligned}
&\gamma(X) = -X, \qquad X \in \{P_+, E_i, J_3\},\\
&\gamma(Y) = -e^{zP_+}Y, \qquad Y \in \{P_-, P_i\},\\
&\gamma(F_1) = -e^{zP_+}(F_1 + zP_-E_1 + zP_2J_3),\\
&\gamma(F_2) = -e^{zP_+}(F_2 + zP_-E_2 - zP_1J_3),\\
&\gamma(K_3) = -e^{zP_+}(K_3 + zP_1E_1 + zP_2E_2);
\end{aligned} \tag{13}
$$

$$
\begin{array}{lll}
[K_3, E_i] = E_i, & [K_3, F_i] = -F_i, & [K_3, J_3] = 0,\\
[J_3, E_i] = -\varepsilon_{ij3}E_j, & [J_3, F_i] = -\varepsilon_{ij3}F_j, & [E_1, E_2] = 0,\\
[E_i, F_j] = \delta_{ij}K_3 + \varepsilon_{ij3}J_3, & [F_1, F_2] = 0, &
\end{array} \tag{14}
$$

$$
[P_\mu, P_\nu] = 0, \qquad \mu, \nu = +, -, 1, 2, \tag{15}
$$

$$
\begin{array}{ll}
[K_3, P_+] = \dfrac{1 - e^{-zP_+}}{z}, & [K_3, P_-] = -P_- - \dfrac{z}{2}(P_1^2 + P_2^2),\\
[K_3, P_i] = (e^{-zP_+} - 1)P_i, & [J_3, P_+] = 0, \qquad [J_3, P_-] = 0,\\
[J_3, P_i] = -\varepsilon_{ij3}P_j, & [E_i, P_-] = P_i, \qquad [E_i, P_+] = 0,\\
[E_i, P_j] = \delta_{ij}\dfrac{1 - e^{-zP_+}}{z}, & [F_i, P_+] = P_i, \qquad [F_i, P_-] = -zP_iP_-,\\
[F_i, P_j] = -zP_iP_j + \delta_{ij}(e^{-zP_+}P_- + \dfrac{z}{2}(P_1^2 + P_2^2)). &
\end{array} \tag{16}
$$

Note that the translation generators $\{P_+, P_-, P_i\}$ define a commutative but non-cocommutative Hopf subalgebra of $U_z(\mathcal{P}(3+1))$ denoted $U_z(\mathcal{T}_4)$, and the Lorentz sector is non-deformed at the algebra level.

5.- Let us consider now the non-deformed Lorentz Hopf algebra $U(so(3,1))$ spanned by the generators $\{E_i, F_i, K_3, J_3\}$ with classical commutation rules (3) and primitive coproduct: $\Delta(X) = 1 \otimes X + X \otimes 1$. We define a right action

$$
\alpha : U_z(\mathcal{T}_4) \otimes U(so(3,1)) \to U_z(\mathcal{T}_4) \tag{17}
$$

as

$$
\alpha(X \otimes Y) \equiv X \triangleleft Y := [X, Y], \qquad X \in \{P_\pm, P_i\}, \quad Y \in \{E_i, F_i, K_3, J_3\}; \tag{18}
$$

explicitly

$$
\begin{array}{lll}
\alpha(P_+ \otimes K_3) = \dfrac{e^{-zP_+} - 1}{z}, & \alpha(P_- \otimes K_3) = P_- + \dfrac{z}{2}(P_1^2 + P_2^2), & \\
\alpha(P_i \otimes K_3) = (1 - e^{-zP_+})P_i, & \alpha(P_+ \otimes J_3) = 0, & \alpha(P_- \otimes J_3) = 0, \\
\alpha(P_i \otimes J_3) = \varepsilon_{ij3} P_j, & \alpha(P_- \otimes E_i) = -P_i, & \alpha(P_+ \otimes E_i) = 0, \\
\alpha(P_i \otimes E_j) = \delta_{ij} \dfrac{e^{-zP_+} - 1}{z}, & \alpha(P_+ \otimes F_i) = -P_i, & \alpha(P_- \otimes F_i) = zP_i P_-, \\
\alpha(P_i \otimes F_j) = zP_i P_j - \delta_{ij}(e^{-zP_+} P_- + \dfrac{z}{2}(P_1^2 + P_2^2)). & &
\end{array}
\tag{19}
$$

Also we define a left coaction

$$
\beta : U(so(3,1)) \to U_z(\mathcal{T}_4) \otimes U(so(3,1)) \tag{20}
$$

by

$$
\begin{array}{l}
\beta(J_3) = 1 \otimes J_3, \qquad \beta(E_i) = 1 \otimes E_i, \\
\beta(F_1) = e^{-zP_+} \otimes F_1 - zP_- \otimes E_1 - zP_2 \otimes J_3, \\
\beta(F_2) = e^{-zP_+} \otimes F_2 - zP_- \otimes E_2 + zP_1 \otimes J_3, \\
\beta(K_3) = e^{-zP_+} \otimes K_3 - zP_1 \otimes E_1 - zP_2 \otimes E_2.
\end{array}
\tag{21}
$$

It can be shown that the right action α and left coaction β just introduced fulfill the compatibility conditions [2] in such manner $(U_z(\mathcal{T}_4), \alpha)$ is a right $U(so(3,1))$-module algebra and $(U(so(3,1)), \beta)$ is a left $U_z(\mathcal{T}_4)$-comodule coalgebra. We summarize the previous discussion in the following theorem, which is the main result of this letter together with the nonlinear basis change (10).

Theorem. *The null-plane quantum Poincaré algebra has the bicrossproduct structure*

$$
U_z(\mathcal{P}(3+1)) = U(so(3,1))^{\beta} \triangleright\!\!\blacktriangleleft_{\alpha} U_z(\mathcal{T}_4). \tag{22}
$$

6.- We would like to stress that the map (10) is invertible, so:

$$
\begin{array}{l}
\widetilde{P}_+ = P_+, \qquad \widetilde{E}_i = E_i, \qquad \widetilde{J}_3 = J_3, \qquad \tilde{z} = z/2, \\
\widetilde{P}_- = e^{zP_+/2} P_-, \qquad \widetilde{P}_i = e^{zP_+/2} P_i, \\
\widetilde{F}_1 = e^{zP_+/2}(F_1 + z(E_1 P_- + J_3 P_2)/2), \\
\widetilde{F}_2 = e^{zP_+/2}(F_2 + z(E_2 P_- - J_3 P_1)/2), \\
\widetilde{K}_3 = e^{zP_+/2}(K_3 + z(E_1 P_1 + E_2 P_2)/2).
\end{array}
\tag{23}
$$

This fact can be applied to reproduce in the bicrossproduct basis the physically relevant operators introduced in [1] such as Casimirs, spin, Hamiltonians and position operators. In particular, the deformed square of the mass M_z^2 is now

$$
M_z^2 = 2P_- \frac{e^{zP_+} - 1}{z} - (P_1^2 + P_2^2) e^{zP_+}, \tag{24}
$$

and the square of the Pauli–Lubanski operator W_z^2 turns out to be

$$
W_z^2 = (W_{13}^z)^2 + (W_{23}^z)^2 + \cosh(zP_+/2)\left(W_+^z W_-^z + W_-^z W_+^z\right) - z^2 M_z^2 (W_+^z)^2/4, \tag{25}
$$

where

$$
\begin{aligned}
W_{i3}^z &= K_3 P_i e^{zP_+} + E_i P_- - F_i \frac{e^{zP_+}-1}{z} + \frac{z}{2}(E_1P_1+E_2P_2)P_i e^{zP_+} \\
&\qquad +(-1)^i J_3 P_{3-i} \frac{e^{zP_+}-1}{2}, \qquad\quad i=1,2, \\
W_-^z &= (F_1P_2 - F_2P_1)e^{zP_+} + J_3 P_- \frac{e^{zP_+}+1}{2} + \frac{z}{2}(E_1P_2 - E_2P_1)P_- e^{zP_+} \\
&\qquad + \frac{z}{2} J_3 (P_1^2 + P_2^2) e^{zP_+}, \\
W_+^z &= (E_1P_2 - E_2P_1)e^{zP_+/2} + J_3 \frac{\sinh(zP_+/2)}{z/2}.
\end{aligned}
\tag{26}
$$

The second order Casimir (24) would give rise to a deformed Schrödinger equation in the same way as in [1, 12], while the Pauli–Lubanski vector (26) would allow to derive quantum Hamiltonians and spin operators. However, it is clear that although the Hopf algebra structure of $U_z(\mathcal{P}(3+1))$ is rather simplified in this new basis, the associated operators adopt a much more complicated form than the original ones.

The map (10) resembles the one given in [13] which allowed to deduce a (factorized) null-plane quantum universal R-matrix: both mappings are related by the interchange $e^{-\tilde{z}\widetilde{P}_+} \leftrightarrow e^{\tilde{z}\widetilde{P}_+}$. Hence, the universal R-matrix reads now

$$
\begin{aligned}
\mathcal{R} = {} & \exp\{zE_2 \otimes e^{zP_+}P_2\} \exp\{zE_1 \otimes e^{zP_+}P_1\} \exp\{-zP_+ \otimes e^{zP_+}K_3\} \\
& \times \exp\{ze^{zP_+}K_3 \otimes P_+\} \exp\{-ze^{zP_+}P_1 \otimes E_1\} \exp\{-ze^{zP_+}P_2 \otimes E_2\}.
\end{aligned}
\tag{27}
$$

Therefore, each basis seems to be useful for a specific purpose and we do not find any privileged basis to express the whole quantum Poincaré algebra together its associated elements (universal R-matrix, quantum Casimirs, etc.).

Finally to mention that the null-plane case in $(2+1)$ dimensions [12] also exhibits this bicrossproduct structure. It looks interesting to profit this bicrossproduct structure of the quantum algebras in order to study by duality the corresponding quantum groups. Work in this direction is in progress and will be published elsewhere.

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References

[1] Ballesteros A, Herranz F J, del Olmo M A and Santander M, 1995 *Phys. Lett. B* **351** 137.

[2] Majid S, 1990 *J. Algebra* **130** 17; *Foundations of quantum group theory* (Cambridge University Press, 1995).

[3] Majid S and Ruegg H, 1994 *Phys. Lett B* **334** 348.

[4] Lukierski J, Nowicky A, Ruegg H and Tolstoy V, 1991 *Phys. Lett. B* **264** 331.

[5] Giller S, Koshiński P, Kunz J, Majewski M and Maślanka P, 1992 *Phys. Lett. B* **286** 57.

[6] Lukierski J, Nowicky A and Ruegg H, 1992 *Phys. Lett. B* **293** 344.

[7] de Azcárraga J A, del Olmo M A, Pérez Bueno J C and Santander M, 1997 *J. Phys. A: Math. Gen.* **30** 3069.

[8] Ballesteros A, Herranz F J, del Olmo M A and Santander M, 1994 *J. Math. Phys.* **35** 4928; 1995 *Lett. Math. Phys.* **33** 273.

[9] de Azcárraga J A and Pérez Bueno J C, 1996 *J. Phys. A: Math. Gen.* **29** 6353.

[10] Ballesteros A, Herranz F J, del Olmo M A, Pereña C M and Santander M, 1995 *J. Phys. A: Math. Gen.* **28** 7113.

[11] Leutwyler H and Stern J, 1978 *Ann. Phys. (N.Y.)* **112** 94.

[12] Ballesteros A and Herranz F J, 1996 *Mod. Phys. Lett. A* **11** 1745.

[13] Ballesteros A, Herranz F J and Pereña C M, 1997 *Phys. Lett. B* **391** 71.